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1986 J. Phys. A: Math. Gen. 19 1093

(http://iopscience.iop.org/0305-4470/19/7/013)

J. Phys. A: Math. Gen. 19 (1986) 1093-1110. Printed in Great Britain

Canonical orthonormal basis for SU(3) ⊃ SO(3): III. Complete set of SU(3) tensor operators

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Received 12 February 1985, in final form 19 July 1985

Abstract. A complete set of tensor operators for SU(3) is given in a model space generated by two Bargmann vectors. Use is made of modified operator patterns to classify these tensors. Their tensorial properties are then discussed in the context of the SU(3) tensor algebra developed by Biedenharn and collaborators. We also give analytical (semi-analytical) expressions for important classes of Wigner coefficients in a Gel'fand (rotational) basis.

1. Introduction

The representation theory of the SU(3) group has been the subject of numerous publications for more than two decades and still continues to attract attention. As a symmetry group in high energy physics and as an approximate symmetry for the nuclear problem, SU(3) belongs to the basic vocabulary of many physicists. Also, from the point of view of representation theory, SU(3) is of prime interest because it is the lowest-dimensional unitary group for which the multiplicity problem arises. It exhibits both an outer multiplicity in the decomposition of the Kronecker product of two unirreps and an inner multiplicity in the classification of states by the group chain SU(3) ⊃ SO(3).

Significant developments clarifying the tensor structure of SU(3) have been given in some recent publications. The outer and inner problems have been shown by Deenen and Quesne (1983, see also Quesne 1984a,b) to reduce to a single problem, the outer one. The latter has theoretically been resolved by Biedenharn and collaborators (see, e.g., Louck 1970) with the use of operator patterns. We will give herein a functional and therefore concrete meaning to this resolution which will be useful in the derivation of SU(3) Wigner and Racah coefficients.

Hassan (1983), using the theory of Weyl invariants, succeeded in deriving an expression for the multiplicity-free coupling coefficients of the direct product $[\lambda_1 0]\otimes[\lambda_2\mu_2]$ of two unitary irreducible representations of SU(3) that does not involve any summation, in contrast to previous expressions that involve up to five summations. However, his expression is quite complicated. We derive here a very simple compact expression for these coefficients using the tensorial approach of Biedenharn and collaborators.

O'Reilly (1982) determined a closed formula for the decomposition of the direct product $[\lambda_1\mu_1]\otimes[\lambda_2\mu_2]$ of two unitary irreducible representations of SU(3) and also determined conditions of existence of $[\lambda_3\mu_3]$ in the decomposition of the product and its multiplicity.

The tensor structure of SU(2) was studied by Schwinger (1965) in a Hilbert space generated by two bosons $(\alpha_+^\dagger, \alpha_-^\dagger)$. It has been shown independently by Biedenharn and Flath (1984) and Bracken and MacGibbon (1984) that a parallel construction can be given for SU(3) in terms of a single irreducible representation of the non-compact group SO(6, 2). To this end, both Biedenharn and Flath (1984) and Bracken and MacGibbon (1984) use a minimal set of two fundamental Bargmann vectors (equivalent to two vector bosons) $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ spanning respectively the fundamental SU(3) unirreps $\{10\}$ and $\{11\}$ to generate their Hilbert space (see also Chacón *et al* 1984). Biedenharn and Flath then proceeded to give the structure of the SU(3) tensor algebra and a classification of all SU(3) tensor operators by decomposing under SU(3) all tensors arising in the enveloping algebra of SO(6, 2). They thereby constructed a generally non-orthogonal but complete set of SU(3) Wigner shift tensors in terms of the fundamental Wigner tensors. It remained, however, to relate the Biedenharn–Flath basis to the elegant classification by operator patterns. The latter step is important for the calculation of SU(3) Wigner and Racah coefficients.

An alternative and elegant realisation of the SO(6, 2) model is given elsewhere (Le Blanc and Rowe 1985d, 1986) in the context of a search for a group structure that would generate a space of lowest weight states for the nuclear symplectic model $Sp(3,\mathfrak{R})$ (Rosensteel and Rowe 1977) as these states are known to carry irreducible representations of the SU(3) group (Rosensteel and Rowe 1980).

Underlying some of these recent results is a very powerful concept in the representation theory of Lie groups, namely the concept of complementarity of two groups embedded in a larger group. To be more precise, use is made of specific unirreps of the larger group, for which the multiplicity problem does not arise, to decompose the Hilbert space with respect to the complementary subgroups. A unique and satisfactory labelling scheme is thereby introduced. Such complementarities have been successfully used, for example, by Biedenharn *et al* (1967) for $\mathscr{U}(n)\otimes U(n)\subset U(n^2)$, Moshinsky and Quesne (1970, 1971) for $O(n)\otimes Sp(d,\mathfrak{R})\subset Sp(dn,\mathfrak{R})$ and Dragt (1965) for $O(2)\otimes SU(3)\subset O(6)$ to classify boson polynomials according to the relevant groups.

More recently, using an idea proposed by Deenen and Quesne (1983, see also Quesne 1984a, b), Le Blanc and Rowe (1985a, part I of this series) used the complementarity of O(3) and $Sp(2,\mathfrak{R})$ in $Sp(6,\mathfrak{R})$ to give canonical orthonormal bases for generic representations $\{h_1h_2\}$ of SU(3) in the group chain $SU(3)\supset SO(3)$. They then proceeded to calculate the reduced matrix elements of the $su(3)\supset so(3)$ algebra in these bases (Le Blanc and Rowe 1985b, part II of this series). A natural extension of this is therefore to give the Wigner and Racah coefficients for SU(3) in the group chain $SU(3)\supset SO(3)$ or, equivalently, to resolve the tensor structure for SU(3).

The purpose of this paper is to indicate how, by extending the complementarity principle to Bargmann tensors as opposed to Bargmann polynomials, one can explicitly construct a complete set of SU(3) tensors and classify them using modified operator patterns. The Hilbert space for our construction is the space of polynomials in two Bargmann vectors $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ (equivalent to two vector bosons). However, in contrast to the Biedenharn–Flath and Bracken–McGibbon constructions, our vectors both carry fundamental $\{10\}$ SU(3) unirreps. We show that a set of tensors is naturally defined on this Hilbert space and the tensors are naturally classified by their transformation properties under the complementary groups $\mathscr{U}(2)$ and SU(3). It will then be shown that the $\mathscr{U}(2)$ labels, which distinguish tensors of the same SU(3) rank, can be identified with (modified) operator patterns, thereby giving, for the first time as far as we are aware, an explicit and concrete realisation of the SU(3) tensors introduced and classified by Biedenharn and co-workers.

We then give some highly illustrative examples of how our results might be used to compute analytically (or semi-analytically) Wigner coefficients in a Gel'fand (or rotational) basis for SU(3). Their computation is a straightforward application of the Wigner–Eckart theorem, an approach which has long been vindicated by Biedenharn and collaborators.

Finally we shall discuss the interpretation of the set of operators constructed in this paper as a complete and canonical set of operators for SU(3) in a direct generalisation of the Schwinger model for SU(2).

2. A model space for SU(3)

In terms of the two Bargmann vectors $g_{\alpha i}$, $\alpha = 1, 2$, $i = 1, 2, 3$, used in I and II of this series, we have a u(3) Lie algebra given (with summation over repeated indices) by

$$C_{ij} = g_{\alpha i}\partial/\partial g_{\alpha j} \tag{2.1}$$

and the Lie algebra of the complementary group $\mathcal{U}(2)$ given by

$$\mathcal{C}_{\alpha\beta} = g_{\alpha i}\partial/\partial g_{\beta i}. \tag{2.2}$$

According to these definitions, the boson vacuum $\langle g|0\rangle = 1$ is seen to carry simultaneously a unirrep $\{000\}$ of U(3) and a unirrep (00) to $\mathcal{U}(2)$. By complementarity, the polynomials in $g_{\alpha i}$ can be decomposed into subsets which carry unirreps $(h_1 h_2)\otimes\{h_1 h_2 0\}$ of the direct product group $\mathcal{U}(2)\otimes\mathrm{U}(3)$. Since the complementarity theorem states that these unirreps are multiplicity free, it follows that a basis of Bargmann polynomials is labelled by the SU(3) labels $\{h_1 h_2\}$ and a set of $\mathcal{U}(2)\otimes\mathrm{SU}(3)$ basis labels.

The simplest, although not necessarily most convenient, basis is given by the Gel'fand bases for both U(3) and $\mathcal{U}(2)$. These are defined by the unirrep labels for the subgroup chains

$$\begin{array}{ccccc} \mathrm{U}(3) & \supset & \mathrm{U}(2) & \supset & \mathrm{U}(1) \\ \{m_{13}m_{23}m_{33}\} & & \{m_{12}m_{22}\} & & \{m_{11}\} \end{array} \tag{2.3}$$

and

$$\begin{array}{ccc} \mathcal{U}(2) & \supset & \mathcal{U}(1) \\ (\gamma_{12}\gamma_{22}) & & (\gamma_{11}) \end{array} \tag{2.4}$$

with the usual betweenness conditions $m_{ij} \geqslant m_{i,j-1} \geqslant m_{i+1,j}$ and $\gamma_{12} \geqslant \gamma_{11} \geqslant \gamma_{22}$. Evidently, for the Bargmann polynomials, one has

$$m_{13} = \gamma_{12} = h_1, \qquad m_{23} = \gamma_{22} = h_2, \qquad m_{33} = 0. \tag{2.5}$$

The physically relevant basis for the su(3) $\supset$ so(3) Lie algebra is given (see I and II) by

$$L_m = -\sqrt{2}[g_\alpha\partial/\partial g_\alpha]^1_m \tag{2.6a}$$

and

$$Q_\nu = \sqrt{6}[g_\alpha\partial/\partial g_\alpha]^2_\nu, \tag{2.6b}$$

where, here and in the following, $[\ \ ]^{()}_{()}$ denotes a tensorial coupling. The bases reducing the corresponding subgroup chain

$$\begin{array}{cccc} \mathrm{SU}(3) \supset & \mathrm{SO}(3) \supset & \mathrm{SO}(2) & \\ \{h_1h_2\}\ \zeta & L & M & , \end{array} \tag{2.7}$$

with $\zeta = (\rho)$ (Le Blanc and Rowe 1985a) or $\zeta = K$ (Elliott 1958) a multiplicity index, will be denoted $|\{h_1h_2\}\zeta LM\rangle$. The SU(3) $\downarrow$ SO(3) reduction for the Bargmann space in six variables was given in I (see also Quesne 1984b).

It will be appropriate in the following to simply use a single index η to label the chosen SU(3) basis and an index ν to label the $\mathscr{U}(2)$ basis. We then have a $\mathscr{U}(2)\otimes\mathrm{U}(3)$ basis of states for the Bargmann space in six variables denoted by

$$\langle g|\{h_1h_2\}\nu\eta\rangle. \tag{2.8}$$

Highest and lowest weight states are defined for U(3) and $\mathscr{U}(2)$ in the usual way. For example, for U(3) we have

$$\begin{aligned} C_{ij}|\{h_1h_2\}\nu\eta_{\mathrm{lw}}\rangle &= 0, \qquad i<j, \\ C_{ij}|\{h_1h_2\}\nu\eta_{\mathrm{hw}}\rangle &= 0, \qquad i>j. \end{aligned} \tag{2.9}$$

In terms of the Bargmann variables, such states are given, for example, by

$$\begin{aligned} \langle g|\{h_1h_2\}\nu = h_1\eta_{\mathrm{lw}}\rangle &= N(h_1, h_2) g_{11}^{h_1-h_2} \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}^{h_2}, \\ \langle g|\{h_1h_2\}\nu = h_1\eta_{\mathrm{hw}}\rangle &= N(h_1, h_2) g_{13}^{h_1-h_2} \begin{vmatrix} g_{13} & g_{12} \\ g_{23} & g_{22} \end{vmatrix}^{h_2}, \end{aligned} \tag{2.10}$$

where

$$N(h_1, h_2) = [(h_1 - h_2 + 1)/(h_1+1)!h_2!]^{1/2}. \tag{2.11}$$

Note that these states are both $\mathscr{U}(2)$ highest weight states as the subspace of $\mathscr{U}(2)$ highest weight states is spanned by the vectors

$$|\{h_1h_2\}\nu = h_1\eta\rangle. \tag{2.12}$$

Within this subspace, every SU(3) unirrep $\{h_1h_2\}$ occurs once and once only. It is therefore, by definition, an SU(3) model space (Bracken and MacGibbon 1984). We now seek a complete set of SU(3) tensor operators that act within this model space.

3. Tensor operators for SU(3)

It has been proved by Biedenharn, Louck and collaborators (see e.g. Louck 1970) that a complete set of SU(3) unit tensor operators can be classified by an upper Gel'fand pattern (commonly referred to as an operator pattern). The components of these unit tensors are denoted

$$\mathscr{T}\left\langle \begin{matrix} & \gamma & \\ \{h_1 & h_2 & 0\} \\ & m & \end{matrix} \right\rangle \tag{3.1a}$$

where

$$\left/\gamma\right\backslash = \left/ \begin{matrix} & \gamma_{11} & \\ \gamma_{12} & & \gamma_{22} \end{matrix} \right\backslash \tag{3.1b}$$

is an operator pattern and

$$\backslash m / = \left\backslash \begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix} \right/ \tag{3.1c}$$

labels a basis for the SU(3) unirrep $\{h\}=\{h_1h_2\}$ according to which this tensor operator transforms. The components of these SU(3) tensor operators are given here in the Gel'fand scheme. However, they may be chosen in any arbitrary way. In particular, it may be convenient in some physical applications to choose a basis which reduces the subgroup chain (2.7). It is again appropriate therefore to use the single index η to label the components of an SU(3) tensor, where η indexes any convenient SU(3) basis. For notational ease, we then denote the components of a general SU(3) unit tensor by

$$\mathscr{T}^{(\gamma)\{h\}}_{\mu\quad\eta}, \tag{3.2}$$

where $(\gamma)=(\gamma_{12}\gamma_{22})$ and $\mu=\gamma_{11}$, rather than by the more cumbersome notation of (3.1). We shall also shortly redefine (γ) and μ in a more convenient way (cf (3.11)).

The labels $\{h\}$ and η define the transformation properties of $\mathscr{T}$ as an SU(3) tensor according to Racah's definition of a tensor operator

$$[C_{\eta'}, \mathscr{T}^{(\gamma)\{h\}}_{\mu\quad\eta}] = \sum_{\eta''} \langle\{h\}\eta''|C_{\eta'}|\{h\}\eta\rangle \mathscr{T}^{(\gamma)\{h\}}_{\mu\quad\eta''} \tag{3.3}$$

where $\langle\{h\}\eta''|C_{\eta'}|\{h\}\eta\rangle$ are matrix elements of the generators of SU(3) between states belonging to a unirrep $\{h\}$ (see e.g. Hecht 1965, Le Blanc and Rowe 1985b). The operator pattern (or, equivalently, the tensor labels γ and μ used here) characterises the remaining tensorial properties of $\mathscr{T}$ of which we mention the most important.

The shifts

$$\Delta = \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} = \begin{pmatrix} \gamma_{11} \\ \gamma_{12}+\gamma_{22}-\gamma_{11} \\ h_1+h_2-\gamma_{12}-\gamma_{22} \end{pmatrix} \tag{3.4}$$

indicate that, when applied to a state belonging to a U(3) unirrep $\{h_1'h_2'0\}$, the tensor $\mathscr{T}$ will map this state to a new unirrep labelled by

$$\{h''\}=\{h'+\Delta\}=\{h_1'+\Delta_1, h_2'+\Delta_2, \Delta_3\}. \tag{3.5}$$

Thus it maps an SU(3) unirrep $\{h_1'h_2'\}$ into $\{h_1'+\Delta_1-\Delta_3, h_2'+\Delta_2-\Delta_3\}$.

Note that the existence of distinct tensors having the same shifts but different operator patterns reflects the existence of the outer multiplicity problem for SU(3). The more technical properties of these operator patterns are thoroughly discussed by Biedenharn and collaborators (see Biedenharn *et al* 1972, Biedenharn and Louck 1972, Louck and Biedenharn 1973 and references therein) to which we refer the reader.

We show in this paper how to construct a complete set of SU(3) tensor operators by restricting a set of $\mathscr{U}(2)\otimes$SU(3) tensor operators defined on the full Bargmann space to the SU(3) model space of $\mathscr{U}(2)$ highest weight states. In attempting to give an explicit construction of the SU(3) tensor operators corresponding to the above operator patterns, it will be useful to modify the latter in a minor but nevertheless very significant way.

First observe that the permissible Biedenharn–Louck operator patterns are the possible unirrep labels for the abstract subgroup chain

$$\begin{array}{ccccc} \mathbf{U}(3) & \supset & \mathbf{U}(2) & \supset & \mathbf{U}(1) \\ \{h_1 h_2 0\} & & (\gamma_{12}\gamma_{22}) & & (\gamma_{11}) \end{array} \tag{3.6}$$

with the canonical embedding of $\mathbf{U}(2) \supset \mathbf{U}(1)$ in $\mathbf{U}(3)$. In this embedding, the U(2) subalgebra is spanned by the subset of $\mathbf{U}(3)$ generators

$$\boldsymbol{C}_{11},\ \boldsymbol{C}_{22},\ \boldsymbol{C}_{12},\ \boldsymbol{C}_{21}. \tag{3.7}$$

Thus the permissible $\mathbf{U}(2)$ labels $(\gamma_{12}\gamma_{22})$ are given by the usual $\mathbf{U}(3) \downarrow \mathbf{U}(2)$ reductions (betweenness conditions)

$$\begin{aligned} &\{100\} \downarrow (10) + (00) \\ &\{200\} \downarrow (20) + (10) + (00) \\ &\{110\} \downarrow (11) + (10) \\ &\{210\} \downarrow (21) + (20) + (11) + (10) \\ &\{310\} \downarrow (31) + (30) + (21) + (20) + (11) + (10) \\ &\quad\vdots \end{aligned} \tag{3.8}$$

Since we are seeking tensor operators for SU(3) rather than U(3), we find it more appropriate to use modified operator patterns in which the permissible patterns are the possible unirrep labels for the subgroup chain

$$\begin{array}{ccccc} \mathbf{SU}(3) & \supset & \mathbf{U}(2) & \supset & \mathbf{U}(1) \\ \{h_1 h_2\} & & (\gamma_1\gamma_2) & & (\nu) \end{array}. \tag{3.9}$$

The difference is that this $\mathbf{U}(2)$ subalgebra is spanned by the subset of $\mathbf{SU}(3)$ generators

$$\boldsymbol{C}_{11} - \boldsymbol{C}_{33},\ \boldsymbol{C}_{22} - \boldsymbol{C}_{33},\ \boldsymbol{C}_{12},\ \boldsymbol{C}_{21}. \tag{3.10}$$

Since the basis states in either scheme are in fact identical, the two sets of operator patterns are trivially related. Indeed, for a $\mathbf{U}(3) \supset \mathbf{U}(2) \supset \mathbf{U}(1)$ Gel'fand state $|\{h_1 h_2 0\}(\gamma_{12}\gamma_{22})\gamma_{11}\rangle$, we have

$$\begin{aligned} \boldsymbol{C}_{33}|\{h_1 h_2 0\}(\gamma_{12}\gamma_{22})\gamma_{11}\rangle &= (h_1 + h_2 - \gamma_{12} - \gamma_{22})|\{h_1 h_2 0\}(\gamma_{12}\gamma_{22})\gamma_{11}\rangle \\ &= \Delta_3|\{h_1 h_2 0\}(\gamma_{12}\gamma_{22})\gamma_{11}\rangle \end{aligned}$$

and hence the $\mathbf{SU}(3)$ labels defined in (3.9) are given by

$$\gamma_1 = \gamma_{12} - \Delta_3, \qquad \gamma_2 = \gamma_{22} - \Delta_3, \qquad \nu = \gamma_{11} - \Delta_3. \tag{3.11}$$

Thus we reinterpret the labels (γ) and μ of the set of tensors (3.2) by $(\gamma) = (\gamma_1\gamma_2)$ and $\nu = \gamma_{11} - \Delta_3$. The permissible $\mathbf{U}(2)$ labels $(\gamma_1\gamma_2)$ are now given by the $\mathbf{SU}(3) \downarrow \mathbf{U}(2)$ reductions

$$\begin{aligned} &\{10\} \downarrow (10) + (-1-1) \\ &\{20\} \downarrow (20) + (0-1) + (-2-2) \\ &\{11\} \downarrow (11) + (0-1) \\ &\{21\} \downarrow (21) + (1-1) + (00) + (-1-2) \\ &\{31\} \downarrow (31) + (2-1) + (10) + (0-2) + (-1-1) + (-2-3) \\ &\quad\vdots \end{aligned} \tag{3.12}$$

and the SU(3) shifts are given directly by

$$\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}=\begin{pmatrix}\Delta_1-\Delta_3\\ \Delta_2-\Delta_3\end{pmatrix}=\begin{pmatrix}\nu\\ \gamma_1+\gamma_2-\nu\end{pmatrix}. \tag{3.13}$$

By modifying the labelling of SU(3) tensors in this way, we shall show in § 4 that it becomes possible to identify the set of SU(3) tensors (3.2) with a set of $\mathcal{U}(2)\otimes\mathrm{SU}(3)$ tensors acting on our Bargmann space in six variables. It should be noted that the Bargmann space contains only U(3) representations $\{h_1h_2h_3\}$ with $h_3=0$. Thus we necessarily have the restriction $\delta h_3=0$ and so the shift δ_3 is redundant.

4. Construction of tensor operators for SU(3)

We first consider the construction of a set of $\mathcal{U}(2)\otimes\mathrm{SU}(3)$ tensors which act on the Bargmann space in six variables. We subsequently restrict their actions to the SU(3) model space and relate them to the complete set of SU(3) unit tensors of § 3.

First, observe that the Bargmann polynomials

$$\langle g|\{h_1h_2\}\nu\eta\rangle \propto T^{(h_1h_2)\{h_1h_2\}}_{\;\;\nu\;\;\;\;\;\;\;\eta}(g) \tag{4.1}$$

can be regarded either as coherent state wavefunctions in a Bargmann space or as multiplicative tensor operators. The polynomials in the differential operators $(\partial/\partial g_{\alpha i})$ are the components of differential tensor operators. By complementarity, one finds that the latter can be decomposed into subsets which carry unirreps

$$(-k_2,-k_1)\otimes\{0,-k_2,-k_1\}$$

of $\mathcal{U}(2)\otimes\mathrm{U}(3)$ and hence unirreps

$$(-k_2,-k_1)\otimes\{k_1,k_1-k_2\}$$

of $\mathcal{U}(2)\otimes\mathrm{SU}(3)$. Thus we obtain the following basic $\mathcal{U}(2)\otimes\mathrm{SU}(3)$ tensors with SU(3) lowest weight components given by

$$\begin{aligned}&T^{(10)\{10\}}_{\;1\;\;\;\mathrm{lw}}(g)=g_{11}, \qquad T^{(10)\{10\}}_{\;0\;\;\;\mathrm{lw}}(g)=g_{21},\\ &T^{(-1-1)\{10\}}_{\;-1\;\;\;\;\mathrm{lw}}(g)=(\partial_1\wedge\partial_2)_1,\end{aligned} \tag{4.2a}$$

and

$$\begin{aligned}&T^{(0-1)\{11\}}_{\;0\;\;\;\;\mathrm{lw}}(g)=\partial/\partial g_{23}, \qquad T^{(0-1)\{11\}}_{\;-1\;\;\;\;\mathrm{lw}}(g)=-\partial/\partial g_{13},\\ &T^{(11)\{11\}}_{\;1\;\;\;\mathrm{lw}}(g)=(g_1\wedge g_2)_3.\end{aligned} \tag{4.2b}$$

Let us now consider the matrix elements of these $\mathcal{U}(2)\otimes\mathrm{SU}(3)$ tensors in the SU(3) model space. Recall that this is the subspace of the Bargmann space spanned by the states $|\{h_1h_2\}\nu=h_1\eta\rangle$. From the Kronecker product rule

$$\{10\}\otimes\{h_1h_2\}=\{h_1+1,h_2\}+\{h_1,h_2+1\}+\{h_1-1,h_2-1\},$$

it follows that the matrix elements

$$\langle\{h_1'h_2'\}h_1'\eta'|T^{(\gamma)\{10\}}_{\;\nu\;\;\;\;\sigma}|\{h_1h_2\}h_1\eta\rangle$$

vanish unless

$$\{h_1',h_2'\}=\{h_1+1,h_2\},\ \{h_1,h_2+1\}\ \text{or}\ \{h_1-1,h_2-1\}.$$

But, from the additive properties of the $\mathcal{U}(1)\subset\mathcal{U}(2)$ label, they also vanish unless

$$h_1'=h_1+\nu.$$

Thus we obtain the selection rules

$$\begin{aligned}
&\langle\{h_1'h_2'\}h_1'\eta'|T^{(10)\{10\}}_{1\ \ \sigma}|\{h_1h_2\}h_1\eta\rangle=0 &&\text{if }\{h_1'h_2'\}\neq\{h_1+1,h_2\},\\
&\langle\{h_1'h_2'\}h_1'\eta'|T^{(10)\{10\}}_{0\ \ \sigma}|\{h_1h_2\}h_1\eta\rangle=0 &&\text{if }\{h_1'h_2'\}\neq\{h_1,h_2+1\},\\
&\langle\{h_1'h_2'\}h_1'\eta'|T^{(-1-1)\{10\}}_{-1\ \ \ \ \sigma}|\{h_1h_2\}h_1\eta\rangle=0 &&\text{if }\{h_1'h_2'\}\neq\{h_1-1,h_2-1\}.
\end{aligned}\tag{4.3}$$

It follows that these tensors have precisely the shifts (3.13) required of them. One also ascertains that the other three basic operators of (4.2) likewise have the prescribed shift properties. The explicit values of the non-vanishing matrix elements will be given in § 5. We thus confirm that the above $\mathcal{U}(2)\otimes\mathrm{SU}(3)$ tensors defined by (4.2) are a complete set of $\{10\}$ and $\{11\}$ SU(3) tensors, when restricted to the model space, with operator patterns given by their $\mathcal{U}(2)\supset\mathcal{U}(1)$ labels.

From the basic tensors (4.2), we can now construct a representative $\mathcal{U}(2)\otimes\mathrm{SU}(3)$ tensor $T^{(\gamma)\{h\}}$, for any $\mathcal{U}(2)$ unirrep (γ) contained in the SU(3) unirrep $\{h\}$ according to the branching rule (3.12), in several ways. For example, we could define the tensor $T^{(\gamma)\{h\}}$ of lowest weight with respect to the group product by

$$T^{(\gamma_1\gamma_2)\{h_1h_2\}}_{\mathrm{lw}\ \ \ \ \mathrm{lw}}=[(T^{(10)\{10\}}_{\mathrm{lw}\ \ \mathrm{lw}})^a(T^{(11)\{11\}}_{\mathrm{lw}\ \ \mathrm{lw}})^b(T^{(-1-1)\{10\}}_{\mathrm{lw}\ \ \ \ \mathrm{lw}})^c(T^{(0-1)\{11\}}_{\mathrm{lw}\ \ \ \mathrm{lw}})^d],\tag{4.4a}$$

with

$$\begin{aligned}
h_1&=a+b+c+d, & a&=\tfrac13h_1-\tfrac23h_2+\tfrac23\gamma_1-\tfrac13\gamma_2,\\
h_2&=b+d, & b&=\tfrac13h_1+\tfrac13h_2-\tfrac13\gamma_1+\tfrac23\gamma_2,\\
\gamma_1&=a+b-c, & c&=\tfrac23h_1-\tfrac13h_2-\tfrac23\gamma_1+\tfrac13\gamma_2,\\
\gamma_2&=b-c-d, & d&=-\tfrac13h_1+\tfrac23h_2+\tfrac13\gamma_1-\tfrac23\gamma_2.
\end{aligned}\tag{4.4b}$$

However, we could equally choose tensors with the basic components ordered differently. One sees from (4.2) that only two components are not commuting and that

$$[T^{(-1-1)\{10\}}(g),\ T^{(11)\{11\}}(g)]_\eta=C_\eta(g)=[g\partial/\partial g]^{(00)\{21\}}_{0\ \ \ \ \eta}.\tag{4.5}$$

It is possible therefore to replace the factors

$$(T^{(11)\{11\}})^e(T^{(-1-1)\{10\}})^e$$

where $e=\min(b,c)$, in (4.4) with the linear combination

$$([T^{(11)\{11\}},\ T^{(-1-1)\{10\}}])^e=(C)^e.\tag{4.6}$$

Thus we define the $\mathcal{U}(2)\otimes\mathrm{SU}(3)$ tensor $T^{(\gamma)\{h\}}$ of lowest weight with respect to the group product by

$$T^{(\gamma_1\gamma_2)\{h_1h_2\}}_{\mathrm{lw}\ \ \ \ \mathrm{lw}}=[(T^{(10)\{10\}}_{\mathrm{lw}\ \ \mathrm{lw}})^a(T^{(11)\{11\}}_{\mathrm{lw}\ \ \mathrm{lw}})^{b-e}(C_{\mathrm{lw}})^e(T^{(-1-1)\{10\}}_{\mathrm{lw}\ \ \ \ \mathrm{lw}})^{c-e}(T^{(0-1)\{11\}}_{\mathrm{lw}\ \ \ \mathrm{lw}})^d],\tag{4.7}$$

where, since $b-e=0$ or $c-e=0$, all the factors commute.

For example, the SU(3) tensors of rank $\{21\}$ are given by

$$\begin{aligned}
T^{(21)\{21\}}_{\nu\ \ \ \ \eta}(g)&=[T^{(10)\{10\}}(g)T^{(11)\{11\}}(g)]^{(21)\{21\}}_{\nu\ \ \ \ \eta}=[g(g_1\wedge g_2)]^{(21)\{21\}}_{\nu\ \ \ \ \eta},\\
T^{(1-1)\{21\}}_{\nu\ \ \ \ \ \eta}(g)&=[T^{(10)\{10\}}(g)T^{(0-1)\{11\}}(g)]^{(1-1)\{21\}}_{\nu\ \ \ \ \ \eta}=[g\partial/\partial g]^{(1-1)\{21\}}_{\nu\ \ \ \ \ \eta},\\
T^{(00)\{21\}}_{0\ \ \ \ \eta}(g)&=[g\partial/\partial g]^{(00)\{21\}}_{0\ \ \ \ \eta}=C_\eta(g),\\
T^{(-1-2)\{21\}}_{\nu\ \ \ \ \ \ \eta}(g)&=[T^{(-1-1)\{10\}}(g)T^{(0-1)\{11\}}(g)]^{(-1-2)\{21\}}_{\nu\ \ \ \ \ \ \eta}\\
&=[(\partial/\partial g_2\wedge\partial/\partial g_1)\partial/\partial g]^{(-1-2)\{21\}}_{\nu\ \ \ \ \ \ \eta}.
\end{aligned}\tag{4.8}$$

We obtain the desirable feature that the components of the tensor $T^{(00)\{21\}}(g)$ are the elements of the su(3) algebra. As another example, the self-conjugate SU(3) tensors of rank $\{2k, k\}$ have a maximal multiplicity set with shift labels $\Delta = (kkk)$; therefore $\delta = (00)$ for which the multiplicity is $k+1$ (Lohe *et al* 1977). According to the above results and as will be seen shortly, these tensors can be obtained by raising in $\mathcal{U}(2)$ from the following set of $\mathcal{U}(2) \otimes \mathrm{SU}(3)$ lowest weight tensors:

$$(T^{(10)\{10\}}_{\mathrm{lw}\ \mathrm{lw}})^{k-i}(C_{\mathrm{lw}})^{i}(T^{(0-1)\{11\}}_{\mathrm{lw}\ \mathrm{lw}})^{k-i}, \qquad i = 0, k. \tag{4.9}$$

We can easily derive the following selection rules for a general tensor $T^{(\gamma)\{h\}}$.

(1) Due to its $\mathcal{U}(2)$ tensorial properties, a tensor $T^{(\gamma)\{h\}}_{\nu\ \eta}$ will map a state belonging to an SU(3) unirrep $\{h'\}$ to a set of SU(3) unirreps $\{h''\}$ such that

$$|j' - j| \leq j'' \leq j' + j \tag{4.10}$$

with

$$j = \tfrac{1}{2}(\gamma_1 - \gamma_2), \qquad j' = \tfrac{1}{2}(h'_1 - h'_2), \qquad j'' = \tfrac{1}{2}(h''_1 - h''_2). \tag{4.11}$$

This mapping is known to be multiplicity free and furthermore satisfies the constraint

$$h''_1 + h''_2 = h'_1 + h'_2 + \gamma_1 + \gamma_2. \tag{4.12}$$

(2) The $\mathcal{U}(2)$ weight and the SU(3) weights are, as usual, additive.

When the space of $\mathcal{U}(2) \otimes \mathrm{SU}(3)$ polynomials is restricted to only lowest weight states with respect to the $\mathcal{U}(2)$ algebra, in order to define a model as in Biedenharn and Flath (1984) or Bracken and MacGibbon (1984), the above selection rules are easily verified to reduce to the ones obtained from the standard operator patterns.

(3) Under restriction to the model subspace, we have from the additivity of the $\mathcal{U}(2)$ weight

$$h''_1 = h'_1 + \nu. \tag{4.13}$$

(4) We then obtain from (4.12) and (4.13) the selection rules

$$h''_1 = h'_1 + \delta_1, \qquad h''_2 = h'_2 + \delta_2, \tag{4.14a}$$

with

$$\delta_1 = \nu, \qquad \delta_2 = \gamma_1 + \gamma_2 - \nu, \tag{4.14b}$$

precisely in accord with (3.13).

(5) As before, the SU(3) weights are additive.

We then conclude that the tensor operators built in this section have both shift properties and multiplicities assigned to them by the operator patterns if their action is restricted to the model space defined at the end of § 2. Note the important fact that, due to the $\mathcal{SU}(3) \downarrow \mathcal{U}(2)$ multiplicity free reduction, tensors belonging to a given multiplicity set carry different $\mathcal{U}(2)$ characters. They can be shown (as briefly discussed in § 5) to form a complete and independent set of SU(3) tensor operators.

5. SU(3) ⊃ SU(2) × U(1) Wigner coefficients for [λ0] coupling

The primary purpose of introducing a complete set of tensor operators for a compact group is to enable one to calculate the Wigner and Racah coefficients for this group. The operators constructed in § 4 can be used to this end. For the sake of completeness, one should first ascertain that the tensors (4.7) form an independent set of tensors. Fortunately, the structure of the tensors (4.7) is such that the arguments used by Draayer and Akiyama (1973) concerning the independence and null space properties of their set of abstract tensors can also be applied to our set of Bargmann tensors (cf also Le Blanc 1985). We will therefore in the following concentrate on specific but highly instructive illustrations of the power of our approach.

We will, here and in § 6, use Elliott's SU(3) quantum labels

$$[\lambda\mu]=[h_1-h_2,\, h_2] \tag{5.1}$$

and his $SU(3)\supset U(1)\times SU(2)$ basis labels $\epsilon\Lambda$. We also occasionally use the usual (angular momentum z projection) label

$$\zeta=\tfrac{1}{2}(2\nu-\gamma_1-\gamma_2) \tag{5.2}$$

for the eigenvalues of the $\mathscr{U}(2)$ weight operator $(\mathscr{C}_{11}-\mathscr{C}_{22})/2$ instead of the $\mathscr{U}(1)\subset\mathscr{U}(2)$ Gel'fand label ν. Finally, we will also use Hecht's (1965) and Draayer and Akiyama's (1973) notation for the SU(3) Wigner and Racah coefficients.

We now state a useful partial result. In our Bargmann space, the matrix element of one of the tensors of (4.7) can be expanded, for a multiplicity free coupling, as

$$\begin{aligned}\langle[\lambda_3\mu_3]\zeta_3\epsilon_3\Lambda_3M_{\Lambda_3}|&T^{(\gamma_1\gamma_2)}_{\zeta_2}{}^{[\lambda_2\mu_2]}_{\epsilon_2\Lambda_2M_{\Lambda_2}}|[\lambda_1\mu_1]\zeta_1\epsilon_1\Lambda_1M_{\Lambda_1}\rangle\\ &=\langle\tfrac{1}{2}\lambda_1\zeta_1;\tfrac{1}{2}\gamma\zeta_2|\tfrac{1}{2}\lambda_3\zeta_3\rangle\langle[\lambda_1\mu_1]\epsilon_1\Lambda_1;[\lambda_2\mu_2]\epsilon_2\Lambda_2\|[\lambda_3\mu_3]\epsilon_3\Lambda_3\rangle\\ &\quad\times\langle\Lambda_1M_{\Lambda_1};\Lambda_2M_{\Lambda_2}|\Lambda_3M_{\Lambda_3}\rangle\langle[\lambda_3\mu_3]\|T^{(\gamma_1\gamma_2)[\lambda_2\mu_2]}\|[\lambda_1\mu_1]\rangle\end{aligned} \tag{5.3}$$

where $\gamma=(\gamma_1-\gamma_2)$ and where the last term is a doubly reduced matrix element under the group product $\mathscr{U}(2)\otimes SU(3)$. Now for a tensor $G^{[\lambda\mu]}=T^{(\lambda+\mu,\mu)[\lambda+\mu,\mu]}$ which is strictly a polynomial in the Bargmann variables (g), we easily deduce from normalisation considerations that its reduced matrix element is given by

$$\langle[\lambda_3\mu_3]\|G^{[\lambda_2\mu_2]}\|[\lambda_1\mu_1]\rangle=N[\lambda_1\mu_1]/N[\lambda_3\mu_3] \tag{5.4}$$

where $N[\lambda\mu]=N(\lambda+\mu,\mu)$ is given by (2.11). Similarly, the reduced matrix element for a tensor $D^{[\lambda\mu]}=T^{(-\lambda,-\lambda-\mu)[\lambda+\mu,\mu]}$ strictly in the derivatives $(\partial/\partial g)$ is easily deduced from (5.4) by Hermiticity considerations. For a multiplicity free coupling, which is the only case considered below, we find

$$\begin{aligned}\langle[\lambda_3\mu_3]\|&D^{[\lambda_2\mu_2]}\|[\lambda_1\mu_1]\rangle\\ &=(-1)^{(\lambda_1+\gamma-\lambda_3)/2}\left(\frac{\lambda_1+1}{\lambda_3+1}\right)^{1/2}(-1)^{\varphi}\left(\frac{\dim[\lambda_1\mu_1]}{\dim[\lambda_3\mu_3]}\right)^{1/2}\frac{N[\lambda_3\mu_3]}{N[\lambda_1\mu_1]}\end{aligned} \tag{5.5a}$$

with (Draayer and Akiyama 1973)

$$\varphi=\lambda_1+\lambda_2-\lambda_3+\mu_1+\mu_2-\mu_3. \tag{5.5b}$$

To illustrate, consider the following matrix element in the Bargmann space:

$$\begin{aligned}\langle[\lambda_1+1\mu_1]\nu_3(=\lambda_1+\mu_1+1)\mathrm{HW}|&T^{(10)[10]}_{1\;\;200}|[\lambda_1\mu_1]\nu_1(=\lambda_1+\mu_1)\mathrm{HW}\rangle\\ &=N[\lambda_1\mu_1]/N[\lambda_1+1\mu_1].\end{aligned} \tag{5.6a}$$

When this result is introduced in (5.3), we deduce that

$$\langle[\lambda_1\mu_1]\mathrm{HW};[10]20\|[\lambda_1+1\mu_1]\mathrm{HW}\rangle=1 \tag{5.6b}$$

as expected for a stretched coupling. All other coefficients pertaining to coupling by one of the [10] SU(3) tensors (4.2) can be obtained similarly. Usually, and for ease of computation, one will deduce the value of an arbitrary SU(3) Wigner coefficient

$$\langle [\lambda_1\mu_1]\epsilon_1\Lambda_1;\, [\lambda_2 0]\epsilon_2\Lambda_2 \| [\lambda_3\mu_3]\epsilon_3\Lambda_3\rangle \tag{5.7a}$$

from a specific coefficient like

$$\langle [\lambda_1\mu_1]\mathrm{HW};\, [\lambda_2 0]\epsilon\Lambda \| [\lambda_3\mu_3]\mathrm{HW}\rangle \tag{5.7b}$$

by using recursion formulae like the ones derived by Hecht (1965). All the SU(3) Wigner coefficients pertaining to coupling by a [10] tensor have been tabulated by Vergados (1968).

In order to demonstrate the versatility and power of our approach, we now consider the (multiplicity free) coupling of states by a tensor carrying a $[\lambda_2 0]$ SU(3) unirrep which, according to (4.7), has an SU(3) highest weight component given by

$$T^{(\lambda_2-2i,-i)[\lambda_2 0]}_{\nu=k-i \quad \mathrm{hw}} = \left(\frac{(\lambda_2-i-k)!}{k!(\lambda_2-i)!}\right)^{1/2} \mathscr{C}^{(k)}_{12}\{(T^{(10)[10]}_{0 \quad \mathrm{hw}})^{\lambda_2-i}\}(T^{(-1-1)[10]}_{-1 \quad \mathrm{hw}})^{i} \tag{5.8}$$

with $0 \leqslant i \leqslant \lambda_2$, $0 \leqslant k \leqslant \lambda_2 - i$ and where

$$\begin{aligned} &\mathscr{C}^{(0)}_{12}\{X\} = X, \qquad &&\mathscr{C}^{(1)}_{12}\{X\} = [\mathscr{C}_{12}, X], \\ &\mathscr{C}^{(2)}_{12}\{X\} = [\mathscr{C}_{12}, [\mathscr{C}_{12}, X]], &&\text{etc.} \end{aligned} \tag{5.9}$$

The tensor (5.8) has, according to (4.14) and (5.1), the SU(3) shifts

$$\begin{aligned} \lambda_3 &= \lambda_1 + \delta_1 - \delta_2 = \lambda_1 - \lambda_2 + i + 2k, \\ \mu_3 &= \mu_1 + \delta_2 = \mu_1 + \lambda_2 - 2i - k, \end{aligned} \tag{5.10a}$$

or, for given $[\lambda_1\mu_1]$ and $[\lambda_3\mu_3]$, is given the following values for i and k:

$$\begin{aligned} i &= [\tfrac{1}{3}(\lambda_1 + \lambda_2 - \lambda_3 + 2\mu_1 - 2\mu_3)], \\ k &= [\tfrac{1}{3}(2\lambda_3 - 2\lambda_1 + \lambda_2 + \mu_3 - \mu_1)]. \end{aligned} \tag{5.10b}$$

Introducing (5.8) into (5.3), we obtain, for $(\epsilon_1\Lambda_1 M_{\Lambda_1})$ and $(\epsilon_3\Lambda_3 M_{\Lambda_3})$ both of highest weight and for states in our model space,

$$\begin{aligned} \langle [\lambda_3\mu_3]\zeta_3 = \tfrac{1}{2}\lambda_3;\, \mathrm{HW} | &T^{(\lambda_2-2i,-i)[\lambda_2 0]}_{(\lambda_3-\lambda_1)/2\ \epsilon_2\Lambda_2 M_{\Lambda_2}} | [\lambda_1\mu_1]\zeta_1 = \tfrac{1}{2}\lambda_1;\, \mathrm{HW}\rangle \\ &= \langle \tfrac{1}{2}\lambda_1 \tfrac{1}{2}\lambda_1;\, \tfrac{1}{2}(\lambda_2 - i)\tfrac{1}{2}(\lambda_3-\lambda_1) | \tfrac{1}{2}\lambda_3 \tfrac{1}{2}\lambda_3\rangle \\ &\quad \times \langle [\lambda_1\mu_1] 2\lambda_1 + \mu_1 \tfrac{1}{2}\mu_1;\, [\lambda_2 0]\epsilon_2\Lambda_2 \| [\lambda_3\mu_3] 2\lambda_3 + \mu_3 \tfrac{1}{2}\mu_3\rangle \\ &\quad \times \langle \tfrac{1}{2}\mu_1 \tfrac{1}{2}\mu_1;\, \Lambda_2 M_{\Lambda_2} | \tfrac{1}{2}\mu_3 \tfrac{1}{2}\mu_3\rangle \langle [\lambda_3\mu_3] \| T^{(\lambda_2-2i,-i)[\lambda_2 0]} \| [\lambda_1\mu_1]\rangle \end{aligned} \tag{5.11}$$

where (see Hecht 1965)

$$\begin{aligned} \epsilon_2 &= 2\lambda_3 + \mu_3 - 2\lambda_1 - \mu_1 \equiv 2\lambda_2 - 3\sigma, \\ \Lambda_2 &= \tfrac{1}{6}(2\lambda_2 - \epsilon_2) = \tfrac{1}{2}\sigma, \\ M_{\Lambda_2} &= \tfrac{1}{2}(\mu_3 - \mu_1). \end{aligned} \tag{5.12}$$

Now, it would seem as though (5.11) is unsuitable to provide us with a means to calculate the SU(3) Wigner (isoscalar) coefficient as there are two unknowns in the right-hand side, namely the SU(3) Wigner coefficient and the $\mathscr{U}(2)\otimes$SU(3) doubly reduced matrix element. But we proceed to prove below that the reduced matrix element is, in the Bargmann space, proportional to the isoscalar coefficient. Furthermore, we derive the constant of proportionality. Thus, there is in fact only one unknown in (5.11), the SU(3) isoscalar factor itself. The overlap matrix on the left-hand side of (5.11) is easily evaluated with the help of techniques similar to the ones used by Moshinsky (1962) for his evaluation of SU(3) boson state normalisation factors.

We now rewrite the reduced matrix element of (5.11) as

$$\begin{aligned}\langle[\lambda_3\mu_3]\| T^{(\lambda_2-2i,-i)[\lambda_2 0]}\|[\lambda_1\mu_1]\rangle &\\ = U([\lambda_2-10][10][\lambda_3\mu_3][\lambda_1\mu_1];\,[\lambda_2 0][\lambda_1\mu_1-1])&\\ \times\langle[\lambda_3\mu_3]\| T^{(\lambda_2-2i+1,-i+1)[\lambda_2-10]}\|[\lambda_1\mu_1-1]\rangle\langle[\lambda_1\mu_1-1]\| T^{(-1-1)[10]}\|[\lambda_1\mu_1]\rangle&\end{aligned}\tag{5.13}$$

where $U([\lambda_2-10]\ldots[\lambda_1\mu_1-1])$ is an SU(3) Racah coefficient (Hecht 1965). From equation (12) of Hecht (1965) and using easily calculated [10] coupling coefficients (or using table 1 from Vergados (1968)), we find

$$\begin{aligned}U([\lambda_2-10][10][\lambda_3\mu_3][\lambda_1\mu_1];\,[\lambda_2 0][\lambda_1\mu_1-1])&\\ = H([\lambda_1\mu_1];\,[\lambda_2 0];\,[\lambda_3\mu_3])&\\ \times\frac{\langle[\lambda_1\mu_1]\text{HW};\,[\lambda_2 0]2\lambda_2-3\sigma\tfrac{1}{2}\sigma\|[\lambda_3\mu_3]\text{HW}\rangle}{\langle[\lambda_1\mu_1-1]\text{HW};\,[\lambda_2-10]2\lambda_2-2-3(\sigma-1)\tfrac{1}{2}(\sigma-1)\|[\lambda_3\mu_3]\text{HW}\rangle}&\end{aligned}\tag{5.14}$$

where

$$\begin{aligned}H([\lambda_1\mu_1];\,[\lambda_2 0];\,[\lambda_3\mu_3]) &= [\tfrac{1}{3}(\lambda_1+\lambda_2+2\lambda_3+2\mu_1+\mu_3+6)]\\ &\times\left(\frac{[\tfrac{1}{3}(\lambda_1+\lambda_2-\lambda_3+2\mu_1-2\mu_3)][\tfrac{1}{3}(\lambda_1+\lambda_2-\lambda_3+2\mu_1+\mu_3+3)]}{\lambda_2(\mu_1+1)(\lambda_1+\mu_1+1)(\lambda_1+\mu_1+2)}\right)^{1/2}.\end{aligned}\tag{5.15}$$

Applying (5.13) recursively, we find

$$\begin{aligned}\frac{\langle[\lambda_3\mu_3]\| T^{(\lambda_2-2i,-i)[\lambda_2 0]}\|[\lambda_1\mu_1]\rangle}{\langle[\lambda_1\mu_1]\text{HW};\,[\lambda_2 0]\epsilon_2\Lambda_2\|[\lambda_3\mu_3]\text{HW}\rangle}&\\ = H([\lambda_1\mu_1];\,[\lambda_2 0];\,[\lambda_3\mu_3])H([\lambda_1\mu_1-1];\,[\lambda_2-10];\,[\lambda_3\mu_3])\ldots&\\ \times H([\lambda_1\mu_1-i];\,[\lambda_2-i0];\,[\lambda_3\mu_3])&\\ \times\langle[\lambda_1\mu_1-1]\| T^{(-1-1)[10]}\|[\lambda_1\mu_1]\rangle\langle[\lambda_1\mu_1-2]\| T^{(-1-1)[10]}\|[\lambda_1\mu_1-1]\rangle\ldots&\\ \times\langle[\lambda_1\mu_1-i]\| T^{(-1-1)[10]}\|[\lambda_1\mu_1-i+1]\rangle&\\ \times\frac{\langle[\lambda_3\mu_3]\| T^{(\lambda_2-i0)[\lambda_2-i0]}\|[\lambda_1\mu_1-i]\rangle}{\langle[\lambda_1\mu_1-i]\text{HW};\,[\lambda_2-i0]\epsilon_2'\Lambda_2'\|[\lambda_3\mu_3]\text{HW}\rangle}.&\end{aligned}\tag{5.16}$$

Since the tensor $T^{(\lambda_2-i0)[\lambda_2-i0]}$ is a polynomial strictly in the Bargmann variables, the ratio on the right-hand side of (5.16) is easily calculated using (5.3) and (5.4). By a straightforward extension of techniques used by Moshinsky (1962), we evaluate an intermediate overlap. We finally find

$$\frac{\langle[\lambda_3\mu_3]\| T^{(\lambda_2-2i,-i)[\lambda_2 0]}\|[\lambda_1\mu_1]\rangle}{\langle[\lambda_1\mu_1]\mathrm{HW}\colon [\lambda_2 0]\epsilon_2\Lambda_2\|[\lambda_3\mu_3]\mathrm{HW}\rangle}$$
$$= \left(\frac{[\frac{1}{3}(\lambda_1+\lambda_2+2\lambda_3+2\mu_1+\mu_3+6)]!}{(\lambda_3+\mu_3+2)(\lambda_3+1)}\right)\{[\tfrac{1}{3}(2\lambda_3+\lambda_1+\lambda_2-\mu_1+\mu_3+3)]!$$
$$\times[\tfrac{1}{3}(\lambda_1+\lambda_2-\lambda_3+2\mu_1-2\mu_3)]![\tfrac{1}{3}(2\lambda_1-\lambda_2+\lambda_3+\mu_1-\mu_3)]!\}^{1/2}$$
$$\times\left(\frac{(\lambda_1+1)[\frac{1}{3}(2\lambda_2-\lambda_1+\lambda_3-2\mu_1+2\mu_3)]![\frac{1}{3}(\lambda_1+\lambda_2-\lambda_3+2\mu_1+\mu_3+3)]!}{(\mu_3+1)\lambda_1!\lambda_2!\lambda_3!(\lambda_1+\mu_1+1)!(\lambda_3+\mu_3+1)![\frac{1}{3}(\lambda_3-\lambda_1-\lambda_2+\mu_1+2\mu_3)]!}\right)^{1/2} \tag{5.17}$$

where the right-hand side is the above-mentioned constant of proportionality between the doubly reduced matrix element and the SU(3) isoscalar coefficient.

Instead of calculating the left-hand side of (5.11) directly, it is actually easier to calculate initially

$$\langle[\lambda_3\mu_3]\nu_3(=\lambda_3+\mu_3)\mathrm{HW}|T^{(\lambda_2-2i,-i)[\lambda_2 0]}_{\nu_2=k-i\ \ \ \mathrm{hw}}\|[\lambda_1\mu_1]\nu_1(=\lambda_1+\mu_1)\epsilon_1\Lambda_1 M_{\Lambda_1}\rangle \tag{5.18}$$

and then to modify the result using (see Hecht 1965)

$$\langle[\lambda_1\mu_1]\epsilon_1\Lambda_1;[\lambda_2 0]\mathrm{HW}\|[\lambda_3\mu_3]\mathrm{HW}\rangle$$
$$=(-1)^{\alpha}\left(\frac{\lambda_2![\frac{1}{3}(2\lambda_1-\lambda_2+\lambda_3+\mu_1-\mu_3)]![\frac{1}{3}(2\lambda_1-\lambda_2+\lambda_3+\mu_1+2\mu_3+3)]!}{\lambda_1!(\lambda_1+\mu_1+1)![\frac{1}{3}(2\lambda_3+\lambda_2-2\lambda_1-\mu_1+\mu_3)]!}\right)^{1/2}$$
$$\times\langle[\lambda_1\mu_1]\mathrm{HW};[\lambda_2 0]\epsilon_2\Lambda_2\|[\lambda_3\mu_3]\mathrm{HW}\rangle \tag{5.19}$$

where $\alpha=[\frac{1}{3}(\lambda_1+\lambda_2-\lambda_3+\mu_3-\mu_1)]$.

We find the following value for the matrix element (5.18):

$$\text{matrix element}=\frac{(-1)^{\alpha}\lambda_3!}{\{[\frac{1}{3}(2\lambda_1-\lambda_2+\lambda_3+\mu_1-\mu_3)]![\frac{1}{3}(\lambda_1+\lambda_2+2\lambda_3+\mu_3-\mu_1+3)]!\}}$$
$$\times\{(\lambda_1+1)![\tfrac{1}{3}(\lambda_1+\lambda_2-\lambda_3+2\mu_1+\mu_3+3)]!$$
$$\times[\tfrac{1}{3}(2\lambda_2-\lambda_1+\lambda_3-2\mu_1+2\mu_3)]!\}^{1/2}$$
$$\times\left(\frac{(\lambda_3+1)(\lambda_3+\mu_3+1)![\frac{1}{3}(2\lambda_1-\lambda_2+\lambda_3+\mu_1-\mu_3)]!}{(\mu_3+1)[\frac{1}{3}(\lambda_3-\lambda_1-\lambda_2+\mu_1+2\mu_3)]![\frac{1}{3}(2\lambda_1-\lambda_2+\lambda_3+\mu_1+2\mu_3+3)]!}\right.$$
$$\left.\times\frac{[\frac{1}{3}(\lambda_1+\lambda_2-\lambda_3+2\mu_1-2\mu_3)]!}{[\frac{1}{3}(2\lambda_3-2\lambda_1+\lambda_2-\mu_1+\mu_3)]!}\right)^{1/2}. \tag{5.20}$$

Introducing (5.17), (5.19) and (5.20) in (5.11), we finally derive the simple and elegant result

$$\langle[\lambda_1\mu_1]\mathrm{HW};[\lambda_2 0]\epsilon_2\Lambda_2\|[\lambda_3\mu_3]\mathrm{HW}\rangle^2$$
$$=\lambda_1!(\lambda_3+1)!(\lambda_1+\mu_1+1)!(\lambda_3+\mu_3+2)!$$
$$\times\{[\tfrac{1}{3}(2\lambda_1-\lambda_2+\lambda_3+\mu_1+2\mu_3+3)]![\tfrac{1}{3}(\lambda_1+\lambda_2+2\lambda_3-\mu_1+\mu_3+3)]!$$
$$\times[\tfrac{1}{3}(2\lambda_1-\lambda_2+\lambda_3+\mu_1-\mu_3)]![\tfrac{1}{3}(\lambda_1+\lambda_2+2\lambda_3+2\mu_1+\mu_3+6)]!\}^{-1} \tag{5.21}$$

which is in agreement with the expressions given in table 2 of Hecht (1965) for

$[\lambda_2 0]=[20]$ and $[40]$. From this coefficient, all other coefficients pertaining to the coupling by the $[\lambda_2 0]$ tensor can be obtained in closed form using the recursion formulae given by Hecht (1965) and Draayer and Akiyama (1973). Hermiticity considerations also lead easily to the Wigner coefficients pertaining to the coupling by a $[0\mu_2]$ tensor.

We have subsequently studied the more complicated case of (generally non-multiplicity free) coupling by a generic $[\lambda\mu]$ SU(3) tensor and given a concrete resolution of the SU(3) multiplicity problem (LeBlanc and Rowe 1985c).

6. SU(3) ⊃ SO(3) Wigner coefficients for [20] coupling: application to the nuclear symplectic model

The nuclear symplectic model $\mathrm{Sp}(3,\mathfrak{R})\supset \mathrm{U}(3)$ has recently been presented (see the recent review by Rowe *et al* (1985)) as a union of independent particle and collective models which admits both superfluid flows and flows with vorticity. Having full regard for nuclear symmetry and the Pauli exclusion principle, the symplectic model is fully compatible with the shell model and hence takes full account of the discrete Fermi nature of finite nuclei.

When computing matrix elements of a rotationally invariant symplectic Hamiltonian, one needs SU(3) ⊃ SO(3) Wigner coefficients corresponding to coupling by a [20] SU(3) tensor since, when decomposed under SU(3), the symplectic algebra contains the [20] symplectic raising operator $A^{[20]}$ and its Hermitian conjugate $B^{[02]}$ in addition to the maximal compact subalgebra U(3). Now, it is of paramount importance to optimise the computation of such coefficients to allow sophisticated shell model calculations within the symplectic framework. The results from parts I and II of this series and of § 5 provide just the right tools for this purpose.

In terms of the SU(3) ⊃ SO(3) basis built in parts I and II (to which we refer the reader for a review of the notation used in this section), the $\mathcal{U}(2)\otimes\mathrm{SO}(3)$ reduced matrix elements of the $l=0,2$ components of a basic [20] SU(3) tensor are defined by

$$\begin{aligned}\langle[\lambda_3\mu_3]\zeta_3(\rho_3)[L_3\varepsilon_3]M_3|\, T^{(\gamma_1\gamma_2)[20]}_{\zeta_2\quad lm}|[\lambda_1\mu_1]\zeta_1(\rho_1)[L_1\varepsilon_1]M\rangle \\ =\langle\tfrac{1}{2}\lambda_1\zeta_1;\tfrac{1}{2}\gamma\zeta_2|\tfrac{1}{2}\lambda_3\zeta_3\rangle\langle L_1M_1;\, lm|L_3M_3\rangle \\ \times\langle[\lambda_3\mu_3](\rho_3)[L_3\varepsilon_3]\|\, T^{(\gamma_1\gamma_2)[20]}_{l}\|[\lambda_1\mu_1](\rho_1)[L_1\varepsilon_1]\rangle.\end{aligned} \tag{6.1}$$

Comparison with equation (5.1), rephrased in terms of the SU(3) ⊃ SO(3) basis, gives immediately the desired isoscalar factors as the ratios

$$\langle[\lambda_1\mu_1](\rho_1)L_1;\,[20]l\|[\lambda_3\mu_3](\rho_3)L_3\rangle=\frac{\langle[\lambda_3\mu_3](\rho_3)[L_3\varepsilon_3]\|\, T^{(\gamma_1\gamma_2)[20]}_{l}\|[\lambda_1\mu_1](\rho_1)[L_1\varepsilon_1]\rangle}{\langle[\lambda_3\mu_3]\|\, T^{(\gamma_1\gamma_2)[20]}\|[\lambda_1\mu_1]\rangle}. \tag{6.2}$$

The $\mathcal{U}(2)\otimes\mathrm{SU}(3)$ reduced matrix element in the denominator is independent of the basis used for SU(3) and can therefore be obtained from (5.17) and (5.21).

The calculation of the numerator is easily performed using techniques extended from part II. First, note that, since the SU(3) unirrep [10] restricts to the SO(3) unirrep [1], the $\mathcal{U}(2)\otimes\mathrm{SU}(3)$ tensors $T^{(10)[10]}$ and $T^{(-1-1)[10]}$ are also irreducible $\mathcal{U}(2)\otimes\mathrm{SO}(3)$ tensors:

$$T^{(10)[10]}_{\zeta\quad 1m}=S^{\{10\}[1]}_{\zeta\quad m},\qquad T^{(-1-1)[10]}_{\zeta\quad 1m}=S^{\{-1-1\}[1]}_{\zeta\quad m}. \tag{6.3}$$

The $\mathcal{U}(2)\otimes\mathrm{SO}(3)$ reduced matrix elements

$$\langle[\lambda_3\mu_3](\rho_3)[L_3\varepsilon_3]\|S^{\{10\}[1]}\|[\lambda_1\mu_1](\rho_1)[L_1\varepsilon_1]\rangle \tag{6.4a}$$

for the canonical $\mathrm{SU}(3)\supset\mathrm{SO}(3)$ basis have been calculated in II. Using similar techniques, we have calculated the reduced matrix elements

$$\langle[\lambda_3\mu_3](\rho_3)[L_3\varepsilon_3]\|S^{\{11\}[1]}\|[\lambda_1\mu_1](\rho_1)[L_1\varepsilon_1]\rangle \tag{6.4b}$$

and give the results in the appendix. The reduced matrix element for the $\mathcal{U}(2)\otimes\mathrm{SO}(3)$ tensor $S^{\{-1-1\}[1]}$ is thereafter given by

$$\begin{aligned}&\langle[\lambda_3\mu_3](\rho_3)[L_3\varepsilon_3]\|S^{\{-1-1\}[1]}\|[\lambda_1\mu_1](\rho_1)[L_1\varepsilon_1]\rangle\\ &\quad=(-1)^{L_1+1-L_3}\left(\frac{2L_1+1}{2L_3+1}\right)^{1/2}\langle[\lambda_1\mu_1](\rho_1)[L_1\varepsilon_1]\|S^{\{11\}[1]}\|[\lambda_3\mu_3](\rho_3)[L_3\varepsilon_3]\rangle.\end{aligned} \tag{6.5}$$

Now note that each [20] $\mathrm{SU}(3)\supset\mathrm{SO}(3)$ tensor can be written (using (5.8) and (6.3)) as a $\mathcal{U}(2)\otimes\mathrm{SO}(3)$ coupled product

$$T_l^{(\gamma_1\gamma_2)[20]}=\langle[10]1;[10]1\|[20]l\rangle,[S^{\{\gamma_1'\gamma_2'\}[1]}S^{\{\gamma_1''\gamma_2''\}[1]}]^{\{\gamma_1\gamma_2\}[l]} \tag{6.6}$$

where $\langle[10]1;[10]1\|[20]l\rangle$ is an $\mathrm{SU}(3)\supset\mathrm{SO}(3)$ isoscalar factor easily derived from (6.3) and (6.4) from part II and (5.4) of this paper.

It is then easily verified that the doubly reduced matrix element in the numerator of (6.2) is given by

$$\begin{aligned}&\langle[\lambda_3\mu_3](\rho_3)[L_3\varepsilon_3]\|T_l^{(\gamma_1\gamma_2)[20]}\|[\lambda_1\mu_1](\rho_1)[L_1\varepsilon_1]\rangle\\ &\quad=\sum_{\lambda\mu\rho L\varepsilon}U\left(\frac{\gamma'}{2}\frac{\gamma''}{2}\frac{\lambda_3}{2}\frac{\lambda_1}{2};\frac{\gamma}{2}\frac{\lambda}{2}\right)\times U(11L_3L_1;lL)\times\langle[10]1;[10]1\|[20]l\rangle\\ &\qquad\times\langle[\lambda_3\mu_3](\rho_3)[L_3\varepsilon_3]\|S^{\{\gamma_1'\gamma_2'\}[1]}\|[\lambda\mu](\rho)[L\varepsilon]\rangle\\ &\qquad\times\langle[\lambda\mu](\rho)[L\varepsilon]S^{\{\gamma_1''\gamma_2''\}[1]}\|[\lambda_1\mu_1](\rho_1)[L_1\varepsilon_1]\rangle\end{aligned} \tag{6.7}$$

enabling us to evaluate the isoscalar factor (6.2).

Since reduced matrix elements of the fundamental tensors (6.3) can be considered as global constants which can be tabulated once and for all, equation (6.2) provides us with an easy and straightforward method for the evaluation of the $\mathrm{SU}(3)\supset\mathrm{SO}(3)$ coefficients needed to perform nuclear symplectic calculations. We expect these developments to reduce by at least an order of magnitude the time required to build numerically any symplectic Hamiltonian matrix.

7. Discussion

Double Gel'fand (Bargmann) polynomials have long been known to offer an effective and economical way of generating multi-rowed representations for the unitary groups. But their applications have been mostly restricted, until very recently, to the construction of $\mathcal{U}(n)\otimes\mathrm{U}(n)$ basis states in the Weyl canonical chain. It became increasingly evident in recent years that they also represent a powerful computational tool for the calculation of some restricted sets of $\mathrm{SU}(n)$ Wigner coefficients (Hassan 1983, Hecht and Suzuki 1983). Furthermore, by invoking the complementarity principle for states belonging to symmetrical representations of the larger groups $\mathrm{Sp}(nd, \mathfrak{R})$, it was realised that they could be used to generate bases for d-rowed representations of $\mathrm{U}(n)$ in the alternative $\mathrm{SU}(n) \supset \mathrm{SO}(n)$ decomposition (Deenen and Quesne 1983, Quesne 1984a, b), thus offering a resolution of the inner multiplicity problem, in addition to providing a natural way of generating unirreps for the non-compact groups $\mathrm{Sp}(d, \mathfrak{R})$ (Rowe *et al* 1985).

In this spirit, we have addressed in the present series of papers the problem of constructing a canonical $\mathrm{SU}(3) \supset \mathrm{SO}(3)$ orthonormal basis for the Bargmann space in six variables by using the complementarity principle for basis states of $\mathrm{O}(3) \otimes \mathrm{Sp}(2, \mathfrak{R})$ in $\mathrm{Sp}(6, \mathfrak{R})$ (part I). We also addressed the problem of calculating matrix elements of the generators of the SU(3) algebra in this basis (part II). These successful undertakings seem to indicate that there are strong possibilities that double Bargmann polynomials could be effectively used to facilitate the calculations of isoscalar factors not only for the compact groups but also for the non-compact groups like $\mathrm{Sp}(n, \mathfrak{R})$, $\mathrm{SO}(n, 2)$ and $\mathrm{SO}^*(2n)$ (in that respect, see Le Blanc and Rowe (1985d, 1986)).

It became increasingly evident to the authors throughout the course of this work that the complementarity principle should, and in fact does, extend to Bargmann tensors in contradistinction to Bargmann polynomials. As it turned out, we have shown that the Biedenharn and Louck classification of SU(3) operators by $\mathscr{U}(2) \subset \mathscr{S}\mathscr{U}(3)$ operator patterns (when suitably modified) leads to a concrete classification of $\mathscr{U}(2) \otimes \mathrm{SU}(3)$ tensors defined on a Bargmann space in six variables and hence offers a group theoretical resolution of the SU(3) outer multiplicity problem in terms of the $\mathscr{U}(2) \otimes \mathrm{SU}(3)$ complementarity. In fact we have shown that the operator patterns acquire a functional meaning related to the existence of the complementary $\mathscr{U}(2)$ group. We have also reduced the calculation of Wigner coefficients to the calculation of matrix elements of concrete tensor operators in Bargmann spaces. The construction of these tensors being based on a rigorous yet simple group theoretical prescription, we may safely assert that the tensors built in this paper correspond to a complete and canonical set of tensor operators for SU(3). The Schwinger model for SU(2) can easily be recast in our formalism (Le Blanc 1985) and we believe that our construction can be considered as a direct generalisation of the Schwinger model. The extension of this construction to $\mathrm{SU}(n)$ tensors, $n \geqslant 4$, should also be straightforward in contrast to the SO(6, 2) model for SU(3) for which no extension seems available.

Because our tensors are simple $\mathscr{U}(2) \otimes \mathrm{SU}(3)$ Bargmann tensors, the calculation of the tensor coupling coefficients can be greatly simplified and (LeBlanc and Rowe 1985c) the algebraic or numerical calculation of all SU(3) Wigner and Racah coefficients facilitated. This is important for sophisticated shell model calculations in an SU(3) basis, such as proposed in the symplectic shell model (Carvalho *et al* 1985), where many such coefficients are required.

Appendix. Doubly reduced matrix elements of the $\mathscr{U}(2) \otimes \mathrm{SO}(3)$ tensor $S^{\{11\}[1]}$

In the canonical $\mathrm{SU}(3) \supset \mathrm{SO}(3)$ basis defined in I and II, we have

$$
\begin{aligned}
&\langle [\lambda_3 \mu_3](\rho_3)[L_3 \varepsilon_3] \| S^{\{11\}[1]} \| [\lambda_1 \mu_1](\rho_1)[L_1 \varepsilon_1] \rangle \\
&\qquad = (-1)^{L_1+1-L_3} F([L_3 \varepsilon_3], [L_1 \varepsilon_1]) \sum_{\rho\rho'} K_{\rho_3 \rho'}([\lambda_3 \mu_3][L_3 \varepsilon_3]) K^{-1}_{\rho\rho_1}([\lambda_1 \mu_1][L_1 \varepsilon_1]) \\
&\qquad\quad \times U(\tfrac{1}{2}\rho_1 \, \tfrac{1}{2}\lambda_3 \, \tfrac{1}{2}\sigma'; \tfrac{1}{2}\rho' \tfrac{1}{2}\sigma')(\rho' \| a^\dagger \| \rho)
\end{aligned}
$$

where the boson reduced matrix element $(\rho'\|a^\dagger\|\rho)$ has been given in I,

$$\sigma' = L_3 - \varepsilon_3$$

and

$$F([L0],[L0]) = -\left(\frac{L+2}{L+1}\right)^{1/2},$$

$$F([L0],[L-1,1]) = \left(\frac{2(L-1)}{(2L+1)(2L-1)}\right)^{1/2},$$

$$F([L1],[L+1,0]) = [2(L+2)]^{1/2},$$

$$F([L1],[L1]) = \left(\frac{L-1}{L}\right)^{1/2}.$$

Also

$$\langle[\lambda_3\mu_3](\rho_3)[L+1,1]\|S^{\{11\}[1]}\|[\lambda_1\mu_1](\rho_1)[L0]\rangle$$
$$= -(L+2)^{1/2}\sum_\rho K_{\rho_3\rho}([\lambda_3\mu_3][L+1,1])K^{-1}_{\rho\rho_1}([\lambda_1\mu_1][L0])$$

while

$$\langle[\lambda_3\mu_3](\rho_3)[L-1,0]\|S^{\{11\}[1]}\|[\lambda_1\mu_1](\rho_1)[L1]\rangle = 0.$$

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